> Project Summary and Outcomes

In this project, students will write a Java method to calculate pi to arbitrary precision. In particular, using Machin's method, students will be able to accurately calculate pi to more than 10,000 digits. They will also get experience in high-precision calculations using the java.Math.BigDecimal class. Finally, they will implement a power series for the arctangent function.

A second focus of this exercise will be testing and program verification and validation. Students will calculate their results against a previously computed decimal expansion of pi. This will lead to a discussion of testing and how it differs from verifying program correctness.

> Student Assessment (see rubric)

How well do students perform each of the following tasks:

- Implement a formula using the BigDecimal class
- Calculate a result with high precision
- Minimize rounding errors
- Return a result with the desired precision
- Validate the results of a calculation

> Prerequisite Knowledge and Skills

Math:

- Pi
- Geometric functions, especially arctangent
- Summations
- Significant digits and rounding

Java:

- Design of methods and support methods
- Return types
- Java packages
- File IO
- Exceptions
- Java API documentation

> This Project Targets the Following Subject Areas(s):

Pre-Java	Java Programming
☐ Hardware Basics	☐ Applet Programming
☒ Software Basics	☐ Application Programming
☐ Networks and Servers	☐ Class Programming
☐ HTML	☒ Method Programming
☐ Action Scripting	
☐ Java Scripting	

> Project-Framing Questions

Essential Question	• How do we calculate pi to a large number of decimal places?
Activity Questions	• What is the history of pi? Why is pi important? • Why is it important to some people to calculate pi to billions of decimal places, even though such precision is not needed for any practical calculation? • Explain the following quote: “Testing can only demonstrate the existence of bugs, not their absence.”
Sample Content Questions	• What are your designs of methods and support methods? • How will you round calculations? • How will you test your results?

> Targeted Content Standards, Benchmarks, or State Frameworks

Massachusetts Curriculum Frameworks:

Mathematics:

- AII.P.8 – Solve a variety of equations and inequalities using

School-to-Career Competencies:

English and Language Arts:

- Oral presentation and discussion

algebraic, graphical, and numerical methods, including the quadratic formula; use technology where appropriate. Include polynomial, exponential, and logarithmic functions; expressions involving absolute values; and simple rational expressions.

- **AII.P.2** – Identify arithmetic and geometric sequences and finite arithmetic and geometric series. Use the properties of such sequences and series to solve problems, including finding the formula for the general term and the sum, recursively and explicitly.

  - Organizing and presenting ideas in a logical order
  - Delivering informal and formal presentations, giving consideration to audience, purpose, and content
  - Identifying elements and organizational structures of effective speeches made for a variety of purposes
  - Composition
  - Collecting information for writing from different texts and sources
  - Writing for different purposes and different audiences
  - Using knowledge of standard English conventions (mechanics, grammar, and spelling) to edit work
  - Writing well-organized stories, essays, or scripts

Social Studies:

  - Being familiar with key historical people, places, events, documents, movements, and other details
  - Demonstrating an understanding of cause and effect, and the relations between events
  - Making connections between key people and events

> Materials and Resources Required for Project

Equipment	• A computer with Java
Consumable Supplies	• Diskettes or CDs
Textbooks/Lesson Guides	• *Java Software Solutions: Foundations of Program Design* by John Lewis and William Loftus. Addison-Wesley, 2005 • *Java Methods*: *An Introduction to Object-Oriented Programming* by Maria and Gary Litvin. Skylight Publishing, 2001. • *Core Java 2, Volume I, – Fundamentals -7th Edition* by Cay Horstmann and Gary Cornell. Sun Microsystems Press, 2005.
Technology	• `Java.Math.BigDecimal class`
Internet Resources	• http://www.joyofpi.com/ • http://mathworld.wolfram.com/MachinsFormula.html

> Activity/Lesson Plan Outline

Overview: For thousands of years, mathematicians have searched for efficient ways to estimate pi. In this project, students will explore various programmatic means of doing this. In the course of completing this project, students will be engaged in history, mathematical, research, writing and coding.

Purpose: Students will learn a number of things in this project. First, they will have a better appreciation for mathematical quantities such as Pi. Second they will do research in the mathematics of pi and in the concept of risk as applied to software development. Third they will get to generate code to implement mathematical algorithms for approximating pi. Fourth, they will prepare written and oral reports of their findings.

Assessment: Assessment will cover four areas: The first will be a written exploration of the history of pi. The second will involve the mathematical correctness of the calculation of pi as implemented in code. Thirdly, the project will involve research and presentation of the concept of risk in software development. Finally assessment will include how well the software performs file input/output.

Pi (π) is the ratio between the circumference of a circle and its diameter. Pi is a fixed value no matter the size of the circle. For millennia, people have tried to approximate this ratio. The ancient Egyptians and Babylonians knew about pi and had approximations for it. The Babylonians knew it was approximately equal to 3. The Egyptians had the approximation formula 4*(8/9)2, which is about 3.160484. (See, for example, and
http://mathforum.org/library/drmath/view/57543.html
http://mathforum.org/dr.math/faq/faq.pi.html.)

Pi is also an irrational number, which means that its decimal representation never terminates or settles into a repeating pattern. In 1897, the Illinois legislature tried to legislate this problem out of existence.

Mathematicians have devised techniques to find better and better decimal approximations of pi. Before computers existed, people calculated pi to hundreds of decimal places by hand. Using computers and sophisticated mathematical techniques, we can now calculate pi to billions of decimal places.

Activity 1: Explore the History of Pi

Why is pi important? Who are the mathematicians who have investigated pi? Why is it important to some people to calculate pi to billions of decimal places, even though such precision is not needed for any practical calculation?
Calculate pi to a high precision using Machin's method. John Machin was a British astronomer and mathematician who presented a formula in 1706 for calculating pi (see, for example, http://en.wikipedia.org/wiki/John_Machin). Machin's method was considered the best way to calculate pi for more than two hundred years.

Machin used the following formula:

```
pi = 16 arctan(1/5) - 4 arctan(1/239)
```

It is then possible to use the Taylor series expansion of arctangent:

$$\arctan(x) = x - x^3/3 + x^5/5 - x^7/7 + ... - (-x)^{2n+1}/(2n+1) + \ldots$$

Brook Taylor, a British mathematician, was a contemporary of Machin's. Taylor series are fundamental to calculus.

Writing and Reflection Prompts: Explore Taylor, Machin, and other mathematicians of the early 18th century. Use the web sites and resources above, and work in teams to develop a written summary of your individual and collective findings. Collaborate with a small group of two to three peers to produce a short, informative essay addressing the following questions:

- Who were the leading mathematicians of the 18th century?
- What problems were they working on?
- What solutions did they find?

- What tools did they use to find these solutions?
- How did mathematicians of the 19th and 20th centuries build upon the work of their 18th-century predecessors?

Activity 2: Perform the Calculation

Overview: In this activity, students will use Machin's method to calculate pi. They will use the `java.Math.BigDecimal` class to perform the high-precision arithmetic needed. This class overcomes the precision limitations of the primitive double type, which can perform accurate calculations to roughly 15 decimal places. The `BigDecimal` class offers a performance-accuracy trade-off: the more precision required, the longer the calculations take, and the more memory is used by a `BigDecimal` object.

The `BigDecimal` class performs calculations accurately at the cost of efficiency. It is possible to implement Machin's method correctly, but the program will run inefficiently, only being able calculate pi to a small number of places in a reasonable amount of time. Therefore, keep efficiency in mind in your designs.

In particular, the `BigDecimal` class is much more efficient when used to do calculations with integer values rather than long decimals. When calculating the Taylor expansion of the arctan function, many multiplications of the operand are involved. When using Machin's method, calculate the arctan of only two values: 1/5 and 1/239. The calculation involving 1/5 does not present an efficiency problem because its decimal expansion is 0.2. However, the value 1/239 has a decimal expansion of 0.0041841004181.

The best way to handle this problem is to create a method `arcTanOfInverse(x)`that will calculate arctan (1/x). Using Machin's method, this function becomes the following:

```
pi = 16 arcTanOfInverse(5) - 4 arcTanOfInverse (239)
```

Then use the Taylor series expansion of arcTanOfInverse:

$$\text{arcTanOfInverse}(x) = 1/x - 1/(3x^3) + 1/(5x^5) - 1/(7x^7) + ... - (-1)^{2n+1}/((2n+1)x^{2n+1}) + \ldots$$

This will be much more efficient to calculate, since you will always be dividing by an integer value.

Tips, Hints, and Tricks: Watch out for rounding errors when doing calculations by noting the following:

- Do all calculations using more precision than required. The `divide` method in the `BigDecimal` class has a scale parameter. Use a scale of five more than needed. For example, if calculating pi to 10,000 places, use a scale of 10,005.
- The `divide` method in the `BigDecimal` class also has a `roundingMode` parameter. Use the constant `BigInteger.ROUND_HALF_EVEN`, which minimizes rounding errors.
- After completing calculations, round to the correct number of digits. Use the `setScale` method in the `BigDecimal` class to eliminate the five extra places.

Another issue is deciding when to stop calculating terms in the arctangent series. The best way to do this is to stop when a term calculates to zero using a given scale.

Activity 3: Give Presentation on Risk

Overview: A major issue in software development is quality assurance (QA), which ensures that the software works correctly. QA is an expensive and major part of the software development process. Students will research perform research and prepare an oral presentation on the concept.

Quality assurance goes hand in hand with good customer service and sound business practices. When a product meets or exceeds client expectations, a company or organization benefits from positive feedback and, hopefully, repeat customers. Conversely, there are consequences in the “real world” for producing technology products that do not meet customer expectations or minimum standards for safety or efficiency. (See, for example, http://www.baselinemag.com/article2/0,1397,1544403,00.asp.) The QA process is a method of reducing such risks and errors before launching a product.

Research and Present: Look at the Risks Forum (http://catless.ncl.ac.uk/Risks). Find an incident in which a software problem presented a major risk. Prepare a short, two-to-three-minute oral presentation about the incident. Answer the following questions in your presentation:

- What happened?
- What were the consequences?
- What were the costs?
- What was the impact of the incident on the organization and on society?
- What could the parties involved have done differently to avoid the incident?

Activity 4: Verify the Implementation

Overview: In this activity, students will determine, and possibly improve, the quality of their program to calculate pi. Here they will verify that their calculations are working correctly.

To do this, compare your results with a result calculated by other technique. See examples of pi calculations at: (http://www.cecm.sfu.ca/). Write a driver class that uses your implementation to calculate 10,000 digits of pi, reads in this file, and compares the two values.

Tips, Hints, and Tricks: There are two ways to read a file that is stored at a web site:

- Read the file directly from the remote location. The following code sets up a `BufferedReader` to read the file from its remote location.

```
String urlName =
    “http://www.cecm.sfu.ca/projects/ISC/dataB/isc/C/pi10000.txt”;
BufferedReader r = new BufferedReader(
    new InputStreamReader(
    (InputStream) (new URL(urlName).getContent())));
```

- Copy the file locally and read it locally.

```
String fileName = “pi10000.txt”;
BufferedReader r = new BufferedReader(new FileReader(fileName));
```

This file is laid out so that the digits of pi are broken into multiple lines. Read through all of the file’s lines, concatenating them together to get a `String` representation of the entire approximation of Pi.

The next step is to use your implementations of Machin’s method to get a decimal representation of pi to the same number of digits and then compare the two values. The easiest way to compare the values is to convert the `BigDecimal` that you calculate to a `String`, using the `toString()` method, and then compare the two Strings character by character using the `charAt() method`.

> Pacing/Timeline

- Activities 1 and 2: 10 hours
- Activity 3: 20 hours
- Activity 4: 20 hours

> Teacher Reflection (For example, what worked well in this project? What would you change if you were to teach it again?)

- What did the students find interesting regarding the history of pi?
- What did the students find interesting regarding the history of mathematics and the Golden Age in the 18th century?
- What were the difficulties in implementing Machin's formula? Were students able to accurately calculate pi?
- What lessons did the students learn about software testing?

> Calculating Pi - Rubric

Attribute	Needs Improvement	Proficient	Advanced	Maximum Pts.
Activity 1: Explore the History of Pi				
Grammar and Style	Makes some grammatical, punctuation, spelling, or usage errors	Makes a few grammatical, punctuation, or spelling errors, although most do not interfere with the overall message or the substance of the text	Writes well. Makes no obvious grammatical or punctuation errors. A pleasure to read.	
Content	Presents little information about the history of pi	Presents some information and facts, showing a basic understanding of the history of pi and its importance	Presents many interesting facts and information in a way that shows a deep understanding of the history and importance of pi	
Sources	Presents little or no research materials. Sources may be dubious in nature.	Cites some substantive Internet sources and texts that appear to be reliable	Cites original materials and Internet sources in ways that demonstrate a thorough search and understanding of the topic	
Activity 2: Perform the Calculation				
Implements a Formula Using the BigDecimal Class	Doesn't use `BigDecimal` or improperly uses `BigDecimal` methods	Calls `BigDecimal` methods properly	Makes good use of constants and common structures	
Calculates a Result With High Precision	Doesn't use the scale parameters properly	Uses the proper scale parameters and implements the	Produces the proper result	

		proper formula	
Minimizes Rounding Errors	Doesn't use the rounding parameters properly	Rounds, but not by using ROUND_HALF_EVEN	Rounds, using ROUND_HALF_EVEN
Returns a Result With the Desired Precision	Doesn't return a result	Returns a result without proper precision	Returns a result with desired precision
Returns a Result With the Desired Precision	Doesn't return a result	Returns a result without proper precision	Returns a result with desired precision

Activity 3: Give Presentation on Risk

Demonstrates Knowledge of Risk	Doesn't understand the studied risk	Understands the studied risk	Understands the studied risk and can make connections to other topics
Demonstrates Good Oral Presentation Skills	Presents poorly. Communicates little knowledge of topic.	Communicates some knowledge of topic clearly and effectively	Communicates clearly and effectively using appropriate body language

Activity 4: Verify the Implementation

Reads the File with the Pre-Calculated Results	Doesn't read the file	Opens and reads through the file	N/A
Validates the Results of a Calculation	Not able to compare results	Compares calculation with file, but results do not match	Compares calculation with file and results match